Laplace Operator or Laplacian

The Laplace operator or Laplacian is which is equal to the divergence[1] of the gradient[1] of a scalar function. It is denoted by the symbol Δ or, more often, ∇^2:

$$\Delta\varphi = \nabla^2\varphi = \nabla.\nabla\varphi$$

The Laplace operator occurs in Laplace's equation as well as many other classical partial differential equations including Poisson's equation, the Helmholtz equation, the wave equation and the diffusion equation.

Definition

Using the definitions of grad and div, we can interpret the Laplacian as a partial differential operator . In applied mathematics, we are interested in the Laplace operator in two or three dimensions. In two dimensions we have:

$$\Delta\varphi = \nabla^2\varphi = \nabla.\nabla\varphi = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} . \begin{pmatrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \end{pmatrix} = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2},$$

and in three dimension,

$$\Delta\varphi = \nabla^2\varphi = \nabla.\nabla\varphi = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} . \begin{pmatrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \\ \frac{\partial \varphi}{\partial z} \end{pmatrix} = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}.$$

Hence the result of the Laplacian operator on a function can be found through differentiation[2].

<u>Example 1</u>

Let $\varphi = x^2 + y^3$, find $\nabla^2\varphi$.

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 2 + 6y.$$

<u>Example 2</u>

Let $\varphi = e^x + \log y + \sin z$, find $\nabla^2\varphi$.

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = e^x - \frac{1}{y^2} - \sin z.$$

[1] Grad Div and Curl
[2] Differentiation

The Laplacian occurs in a number of standard partial differential equations[3] such as the Laplace equation, the diffusion/heat conduction equation, the Poisson equation and the Wave equation/Helmholtz equation.

[3] Partial Diferential Equations